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Gallai's conjecture for disconnected graphs

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Abstract

The path number $p(G)$ of a graph G is the minimum number of paths needed to partition the edge set of G. Gallai conjectured that $p(G) \leqslant \lfloor (n+1)/2 \rfloor$ for every connected graph G of order n. Because the graph consisted of disjoint triangles, the best one could hope for in the disconnected case is $p(G) \leqslant \lfloor \frac{2}{3}n \rfloor$. We prove the sharper result that $p(G) \leqslant \frac{1}{2}u + \lfloor \frac{2}{3}g \rfloor$ where u is the number of odd vertices and g is the number of nonisolated even vertices.

1. Introduction

Only simple, finite, undirected graphs are considered in this paper. A vertex is called *odd* or *even* depending on whether its degree is odd or even, respectively. A graph in which every vertex is even is called an *even graph*. A *eulerian graph* is a connected even graph. A *decomposition* of a graph $G=(V,E)$ is a partition of the edge set E of G into subgraphs. Every graph is decomposable into paths, and every even graph is decomposable into cycles. The *path number* $p(G)$ of is the minimum number of paths needed for a path decomposition, and the *cycle number* $cy(G)$ is the minimum number of cycles required for a cycle decomposition.

The written history of these ideas begins with the paper [8] by Lovász. Gallai conjectured that, for any connected graph G, $p(G) \leqslant \lfloor (n+1)/2 \rfloor$ where n is the *order* of G. In transforming a path-cycle decomposition into a path decomposition, the key idea of Lovász was to partition each cycle into two paths. Donald improved this idea by partitioning each pair of cycles into three paths. Our approach is based on partitioning of each triple of cycles into at most four paths. This is described in greater detail in Section 2. Several results exist for special families of graphs [2,4–7,11], but for general (possibly disconnected) graphs the main results are due to Lovász and Donald.

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Since we are concerned with graphs which are posibly not connected, we use $n(G)$ for the number of nonisolated vertices in G. Let $u(G)$ and $g(G)$ denote the number of *odd* and *nonisolated even* vertices in G, respectively. The operand of these functions may be dropped when the meaning is clear.

Theorem A (Lovász [8]). *For every graph G,*

1. *G is decomposable into at most $n/2$ paths and cycles.*
2. $p(G)\leqslant n-1$.
3. *If* $g(G)=0$, *then* $p(G)=\frac{1}{2}n$. *Lemma* 2.4
4. *If* $g(G)\geqslant 1$, *then* $p(G)=u/2+g-1$.

Theorem B (Donald [3]). 1. *For any graph G,* $p(G)\leqslant\frac{3}{4}n$.
2. *For any graph G,* $p(G)\leqslant u/2+\lfloor\frac{3}{4}g\rfloor$.

Because the graph consisted of disjoint triangles, the best we can hope for in the disconnected case is $p(G)\leqslant\lfloor\frac{2}{3}n\rfloor$. In Section 3 we prove the following theorem which generalizes all of these results.

Theorem 1. *For any graph G (possibly disconnected),* $p(G)\leqslant u/2+\lfloor\frac{2}{3}g\rfloor$.

Our proof uses the following theorem of Pyber [9].

Theorem C (Pyber [9]). *If each cycle of a graph G contains an odd vertex, then* $p(G)\leqslant n/2$.

Note that Kouider and Lonc proved

Theorem D (Kouider–Lonc [7]). *Let H be any 2k regular graph with girth g and* $2k\leqslant 2g-3$. *Then H is decomposable into n/2 paths of equal length.*

If $Q\subseteq R\subseteq G$, we define $\bar{N}_R(V(Q))$ to be the neighbors of $V(Q)$ in R that are not vertices of Q; that is, $\bar{N}_R(V(Q))=N_R(V(Q))-V(Q)$. For any vertex x of R, let $d_R(x)=|N_R(x)|$.

2. From cycles to paths

Lemma 2.1. *Let G_r be a graph decomposable into r cycles and at most two edges, all containing the vertex x, for* $r=1,2,3$. *Then* $p(G_r)=r+1$.

Proof. Assume that G is a counterexample. Call the cycles C_1,C_2,C_3. By adding extra pendent edges at x we can assume that exactly two edges xu,xv are used in the decomposition. Let $xx_i\in E(C_i)$ for $i=1,2,3$.

Case 1: $r = 1$. Let $xw \in E(C_1)$. If $u \notin V(C_1)$, then the paths wxv, $C_1 - xw + xu$ decompose G_1. Thus, $u, v \in V(C_1)$. There must be an edge uz of C_1 such that $z \neq v$. Then the paths $zuxv$, $C_1 - zu$ decompose G_1.

Case 2: $r = 2$. Assume that $u \notin V(C_1)$. If $v \notin V(C_2)$, the paths x_1xx_2, $C_1 - xx_1 + xu$, $C_2 - xx_2 + xv$ decompose G_2. Hence, $v \in V(C_2)$. There must be an edge vw of C_2 such that $w \neq x_1$. Then the path x_1xvw contains an edge from each cycle and thereby yields a decomposition of G_2 into 3 paths, a contradiction. It follows that u lies in C_1, and so by symmetry both vertices u, v lie in both cycles C_1, C_2. There must be at least one neighbor u_1 of u on C_1 and a neighbor v_2 on C_2 so that u_1uxvv_2 is a path. Since it contains uxv and an edge from each cycle, again we have a decomposition into 3 paths, a contradiction.

Case 3: $r = 3$. If $u \notin V(C_i)$ for some $i \in \{1, 2, 3\}$, then $(C_i - xx_i) + xu$ is a path. By Case 2, the rest of the graph is decomposable into 3 paths. So for each $i = 1, 2, 3$, we have $u \in V(C_i)$ and analogously $v \in V(C_i)$; hence, $d(u) = d(v) = 7$.

If $uv \in E(G)$, we can suppose it is an edge of C_1.

Let $uy \in E(C_1)$ where $y \neq v$. If $y \notin V(C_i)$ for some $i \in \{2, 3\}$, then for any $uy_i \in E(C_i)$ it follows that $C_i - uy_i + uy$ is a path. Since we have two choices for an edge vy_j of $C_j (j \notin \{1, i\})$ incident with v, we have another path y_iuxvy_j. This gives a decomposition into 4 paths, a contradiction. Hence, $d(y) = 6$, and y is incident with at least 2 edges of $C_2 \cup C_3$ which are incident with neither x nor v.

At v there exists at least 3 edges which are not incident with uy and are not contained in C_1, and so two of these must lie in the same cycle, say C_2. Choose an edge $yy' \in E(C_3)$ where $y' \notin \{x, v\}$, if possible. Of the two choices available at v choose an edge $vv' \in E(C_2)$ so that $y' \neq v'$. Then the path $y'yuxvv'$ contains exactly one edge from each cycle, and the rest of the graph is decomposable into 3 paths, a contradiction. Thus, it is not possible to choose an edge $yy' \in E(C_3)$ as desired; that is, $xy, yv \in E(C_3)$. Select another edge $vv'' \in E(C_3)$. Choose a neighbor $y'' \neq v''$ of y in C_2. Then the path $y''yuxvv''$ contains exactly one edge from each cycle, and the rest of the graph is decomposable into 3 paths, the final contradiction. □

Lemma 2.2. *If* $cy(G) = \Delta(G)/2 \leqslant 3$, *then* $p(G) \leqslant cy(G) + 1$.

Proof. This follows immediately from Lemma 2.1. □

Lovász's construction about the vertex x is defined in [3] as follows. Suppose x is a vertex of a graph G, $Z = \{a_1, a_2 \ldots, a_k\}$ is a nonempty set of neighbors of x, $G' = G - E(x, Z)$, and Σ' is a path decomposition of G'. Suppose also that every neighbor of x in G begins some path of Σ'. For each integer $i \in \{1, 2, \ldots, k\}$, we define a *Lovász sequence* $S_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,r_i})$ with the property that each $a_{i,\mu}$ is a neighbor of x in G. Let $a_{i,0} = a_i$. As defined, each $a_{i,\mu}$ is the end of at least one path $U_{i,\mu}$ of Σ'. If $U_{i,\mu}$ does not contain x, end the sequence. If x lies on $U_{i,\mu}$, let $a_{i,\mu+1}$ be the last vertex on $U_{i,\mu}$ before reaching x. Now, we define a function $f : \Sigma' \to \Sigma$ that maps each path of Σ' to a path or cycle of G. Let $U \in \Sigma'$. If U is not a $U_{i,\mu}$, let

$f(U)=U$. Otherwise, $U=U_{i,\mu}$ for some integers i, μ, and we modify each U based on the following cases:

(a) U contains x.
(a′) U does not contain x.
(b) $U=U_{i,v}=U_{j,v}$ where $(i,\mu)\neq(j,v)$.
(b′) $U=U_{i,v}$ and the labeling of U is unique.

Since (a) and (a′) are mutually exclusive and (b) and (b′) are mutually exclusive, we complete the construction with the following four mutually exclusive cases:

Case (ab): Let $f(U)=U+xa_{i,\mu}+xa_{j,v}-xa_{i,\mu+1}+xa_{j,v+1}$.
Case (ab′): Let $f(U)=U+xa_{i,\mu}-xa_{i,\mu+1}$.
Case (a′b): Let $f(U)=U+xa_{i,\mu}+xa_{j,v}$.
Case (a′b′): Let $f(U)=U+xa_{i,\mu}$.

This completes the definition of f and Lovász's construction. More precisely, we call this the (x,G',G)-construction because we vary these parameters frequently in our proof of Theorem 1. Lovász's construction was first used to prove in some sense the following lemma in [8] and then to prove it more explicitly in [3].

Lemma 2.3. *Suppose x is a vertex of a graph G, Z is a nonempty set of neighbors of x, $G'=G-E(x,Z)$, and Σ' is a path (or a path-cycle) decomposition of G'. If every neighbor of x in G begins some path of Σ', then the (x,G',G)-construction produces a path-cycle decomposition Σ of G where $|\Sigma|=|\Sigma'|$ and every cycle of Σ contains x.*

Recall that k is the number of edges incident with x which are deleted from G to make G'; that is, $k=|Z|$ in the previous lemma. For the proof of Theorem 1 in Section 3 it is convenient to let k be the number of even neighbors of x.

Lemma 2.4. *Suppose we perform the (x,G',G)-construction to obtain a path-cycle decomposition Σ of G. Then the number q of cycles in Σ satisfies $0\leqslant q\leqslant\frac{1}{2}k$. Morever, $p(G)\leqslant p(G')+\lfloor\frac{1}{3}(q+2)\rfloor$.*

Proof. The first part of the lemma was explained by Donald [3]. For the second part we partition the cycles into $\lceil q/3\rceil$ sets of size 3, except the last set may have size 1 or 2. We produce a path decomposition by decomposing each set of i cycles $(i=1,2,3)$ into $i+1$ paths according to Lemma 2.2. □

Lemma 2.5. *If $k=1$, then the (x,G',G)-construction produces a path-cycle decomposition Σ of G where at least one path of Σ begins at x.*

Proof. The last element $a_{1,r}$ of the sequence S_1 begins a path $U=U_{1,r}$ which does not contain x. Since $k=1$, U is of type (a′b′) and so x is an end of the path $f(U)$. □

Lemma 2.6. *Let G be a graph decomposable into two cycles C_1, C_2 and an edge xx^* such that C_1 contains x (and possibly x^*) and C_2 contains x^* (and possibly x). Then $p(G) \leqslant 3$.*

Proof. By Lemma 2.1 we can suppose that $x^* \notin C_1$and $x \notin C_2$. Let u be a neighbor of x in C_1, and let v be a neighbor of x^* in C_2 which is different from u. Then $G - E(uxx^*v)$ consists of two paths. □

3. Proof of Theorem 1

Let us denote by $m(G)$ the size of a graph G. The proof is by contradiction. Let G be a counterexample for which the parameter $\lambda(G) = 2m(G) - n(G)$ is as small as possible. If $\lambda(G) \leqslant 0$, then G is a collection of disjoint edges, possibly with some isolated vertices. In this case the conclusion follows immediately, and so we must have $\lambda(G) = \lambda \geqslant 1$.

Claim 3.1. Every component of G contains a cycle of even vertices.

Proof. Assume not. Then Theorem C implies G is not connected. Let G_1 be a component of G that contains no cycle of even vertices. Let $G_2 = G - E(G_1)$. Applying Theorem C to G_1 and the induction hypothesis to G_2 we have

$$p(G) = p(G_1) + p(G_2) \leqslant \frac{u}{2} + \left\lfloor \frac{1}{2} g(G_1) \right\rfloor + \left\lfloor \frac{2}{3} g(G_2) \right\rfloor \leqslant \frac{u}{2} + \left\lfloor \frac{2}{3} g \right\rfloor,$$

a contradiction. □

Let two adjacent even vertices x, x^* be chosen arbitrarily from G. Let $a_1, a_2, \ldots, a_k$ be the even vertices adjacent to x with $a_k = x^*$.

Claim 3.2. $k \not\equiv 0, 1, 5 \bmod 6$.

Proof. Suppose $k \equiv 0, 1, 5 \bmod 6$. Let $G' = G - xa_1 - xa_2 - \cdots - xa_k$. Since G is a minimum counterexample, $p(G') \leqslant \frac{1}{2}u(G') + \lfloor \frac{2}{3} g(G') \rfloor$. From Lemma 2.4 exactly q cycles are produced by the (x, G', G)-construction, and we have the inequalities $0 \leqslant q \leqslant \frac{1}{2}k$ and

$$p(G) \leqslant \tfrac{1}{2}u(G') + \lfloor \tfrac{2}{3} g(G') \rfloor + \lfloor \tfrac{1}{3}(q+2) \rfloor. \tag{1}$$

If $k \equiv 0 \bmod 6$, vertex x is even in G'. Since all even neighbors of x are odd in G', $u(G') = u(G) + k$ and $g(G') \leqslant g(G) - k$. Also $2q \leqslant k$, and so $\lfloor \frac{1}{3}(q+2) \rfloor \leqslant \frac{1}{6}k$. Hence,

$$p(G) \leqslant \tfrac{1}{2}(u(G) + k) + \left\lfloor \tfrac{2}{3}(g(G) - k) \right\rfloor + \tfrac{1}{6}k = \tfrac{1}{2}u(G) + \lfloor \tfrac{2}{3} g(G) \rfloor.$$

If $k \equiv 1$ or $5 \bmod 6$, x is odd in G'. So $u(G') = u(G) + k + 1$ and $g(G') = g(G) - k - 1$. Since k is odd, $2q \leqslant k - 1$ and

$$\left\lfloor \frac{1}{3}(q+2) \right\rfloor \leqslant \begin{cases} \frac{1}{6}(k-1) & \text{if } k \equiv 1 \bmod 6, \\ \frac{1}{6}(k+1) & \text{if } k \equiv 5 \bmod 6. \end{cases}$$

In each case we employ Inequality (1). If $k \equiv 1 \bmod 6$,

$$\begin{aligned} p(G) &\leqslant \tfrac{1}{2}(u(G) + k + 1) + \left\lfloor \tfrac{2}{3}(g(G) - k - 1) \right\rfloor + \tfrac{1}{6}(k-1) \\ &= \tfrac{1}{2}u(G) + \left\lfloor \tfrac{2}{3}g(G) - \tfrac{1}{3} \right\rfloor \leqslant \tfrac{1}{2}u(G) + \left\lfloor \tfrac{2}{3}g(G) \right\rfloor. \end{aligned}$$

If $k \equiv 5 \bmod 6$, then

$$\begin{aligned} p(G) &\leqslant \tfrac{1}{2}(u(G) + k + 1) + \left\lfloor \tfrac{2}{3}(g(G) - k - 1) \right\rfloor + \tfrac{1}{6}(k+1) \\ &= \tfrac{1}{2}u(G) + \left\lfloor \tfrac{2}{3}g(G) \right\rfloor. \qquad \square \end{aligned}$$

Claim 3.3. $k \not\equiv 2 \bmod 6$.

Proof. Suppose the contrary. Let $G_1 = G + ya_k$ where y is a new vertex. Then $p(G) \leqslant p(G_1)$, and it suffices to show that $p(G_1) \leqslant \frac{1}{2}u(G) + \lfloor \frac{2}{3}g(G) \rfloor$. Let $G_1' = G - xa_1 - xa_2 - \cdots - xa_{k-1}$. Since $k - 1 \geqslant 1$, at least one edge is deleted from G_1', and x is not isolated in G_1'. Now $\lambda(G_1') = 2(m(G) - k + 2) - (n(G) + 1) = \lambda(G) - (2k - 3) < \lambda$. Every vertex adjacent to x in G_1' is odd, and, hence, begins some path in any path decomposition of G_1'. Perform the (x, G_1', G_1)-construction, and apply Lemma 2.4. Only $k - 1$ edges were deleted, and hence the number q of cycles created must satisfy $2q \leqslant k - 1$. Since $k \equiv 2 \bmod 6$, x is odd in G_1'. Hence, $u(G_1') = u(G) + k + 2$ and $g(G_1') = g(G) - k - 1$. Also, $\lfloor \frac{1}{3}(q+2) \rfloor = \lfloor (2q+4)/6 \rfloor \leqslant \lfloor \frac{1}{6}(k+3) \rfloor = \frac{1}{6}(k-2)$. From Lemma 2.4 we get

$$\begin{aligned} p(G_1) &\leqslant \tfrac{1}{2}(u(G) + k + 2) + \left\lfloor \tfrac{2}{3}(g(G) - k - 1) \right\rfloor + \tfrac{1}{6}(k-2) \\ &= \tfrac{1}{2}u(G) + \left\lfloor \tfrac{2}{3}g(G) \right\rfloor. \qquad \square \end{aligned}$$

Claim 3.4. $k \in \{3, 4\}$.

Proof. Suppose not. Then $k \geqslant 9$. Also suppose first that $k \equiv 3 \bmod 6$. Let $G_2 = G - xa_1 - xa_2$. Then the vertices a_1, a_2 are odd in G_2, and so $u_2 = u + 2$ and $g_2 = g - 2$. Let $G_2' = G_2 - xa_3 - \cdots - xa_k$. Now $u_2' = u_2 + k - 1 = u + k + 1$ and $g_2' = g_2 - k + 1 = g - k - 1$. Using a minimum path decomposition Σ_2' of G_2' we perform Lovász's construction about x in G_2' with respect to G_2, and this yields a path-cycle decomposition Σ_2 of G_2 having a certain number q of cycles. Applying Lemma 2.4 we have $p(G_2) \leqslant p(G_2') + \lfloor \frac{1}{3}(q+2) \rfloor$. If $q > 0$, we have produced a path decomposition by decomposing sets of i cycles ($i = 1, 2, 3$) into $i + 1$ paths. In one of these cases instead now we decompose one set of i cycles together with the edges xa_1, xa_2 into $i + 1$ paths according to Lemma 2.1.

Thus, $p(G) \leqslant p(G_2') + \lfloor \frac{1}{3}(q+2) \rfloor = \frac{1}{2}(u(G)+k+1) + \lfloor \frac{2}{3}(g(G)-k-1) \rfloor + \frac{1}{6}(k-3) = \frac{1}{2}u(G) + \lfloor \frac{2}{3}g(G) - \frac{2}{3} \rfloor$, a contradiction. Hence, $q=0$. In other words, no cycles are produced by Lovász's construction, and Σ_2 is a path decomposition of G_2. Using one additional path to cover the edges xa_1, xa_2 we have $p(G) \leqslant |\Sigma_2| + 1 = |\Sigma_2'| + 1 \leqslant \frac{1}{2}(u(G)+k+1) + \lfloor \frac{2}{3}(g(G)-k-1) \rfloor + 1 = \frac{1}{2}u(G) + \lfloor \frac{2}{3}g(G) - k/6 + \frac{5}{6} \rfloor$, a contradiction.

It follows that $k \equiv 4 \bmod 6$. Let $G_3 = G - xa_1 - xa_2 + ya_k$ where y is a new vertex. Then the vertices a_1, a_2, a_k, y are odd in G_3, and so $u_3 = u+4$ and $g_3 = g-3$. Let $G_3' = G - xa_3 - \cdots - xa_{k-1}$. Now $u_3' = u_3 + k - 2 = u + k + 2$ and $g_3' = g - k - 1$. Proceeding as above we use a minimum path decomposition Σ_3' of G_3' to perform the (x, G_3', G_3)-construction, and get a path-cycle decomposition Σ_3 of G_3 with q cycles. Lemma 2.4 gives $p(G_3) \leqslant p(G_3') + \lfloor \frac{1}{3}(q+2) \rfloor$. If $q>0$, we can transform one set of i cycles together with the edges xa_1, xa_2 into $i+1$ paths according to Lemma 2.1. The remaining edge ya_k is simply removed. Thus, $p(G) \leqslant p(G_3') + \lfloor \frac{1}{3}(q+2) \rfloor = \frac{1}{2}(u(G)+k+2) + \lfloor \frac{2}{3}(g(G)-k-1) \rfloor + \frac{1}{6}(k-4) = \frac{1}{2}u(G) + \lfloor \frac{2}{3}g(G) - \frac{1}{3} \rfloor$, a contradiction. Hence, Σ_3 is a path decomposition of G_3. Adding the path a_1xa_2 to the decomposition we have $p(G) \leqslant |\Sigma_3| + 1 = |\Sigma_3'| + 1 \leqslant \frac{1}{2}(u(G)+k+2) + \lfloor \frac{2}{3}(g(G)-k-1) \rfloor + 1 = \frac{1}{2}u(G) + \lfloor \frac{2}{3}g(G) - \frac{k}{6} + \frac{4}{3} \rfloor$, a contradiction. □

Let R be the subgraph induced by the even vertices of G, and let $t(xx^*) = |N_R(x) \cap N_R(x^*)|$. Then $t(xx^*)$ is the number of triangles of R containing the edge xx^*.

Claim 3.5. $1 \leqslant t(xx^*) \leqslant 3$.

Proof. Claim 3.4 implies $t(xx^*) \leqslant 3$. Assume $t(xx^*) = 0$. Let $G_0 = G - xx^*$. Let Z be the set of even neighbors of x in G_0 and $G_1 = G_0 - E(x, Z)$. Let Z^* be the set of even neighbors of x^* in G_1 and $G_2 = G_1 - E(x^*, Z^*)$. From Claim 3.4 applied to x and x^* it follows that $u(G_2) = u(G) + 6$ and $g(G_2) \leqslant g(G) - 6$. Let Σ_2 be a minimum path decomposition of G_2. Then

$$|\Sigma_2| \leqslant \tfrac{1}{2}(u(G)+6) + \lfloor \tfrac{2}{3}(g(G)-6) \rfloor \leqslant \tfrac{1}{2}u(G) + \lfloor \tfrac{2}{3}g(G) \rfloor - 1.$$

Let Σ_1 be the path-cycle decomposition of G_1 obtained from Σ_2 by performing the (x^*, G_2, G_1)-construction. Construct a path-cycle decomposition $\sum_0$ of G_0 by performing the (x, G_1, G_0)-construction. Lemma 2.3 implies $|\Sigma_0| = |\Sigma_1| = |\Sigma_2|$. Observe that at most two cycles appear in Σ_0, one containing x (and possibly x^*) and one containing x^* (and possibly x). If no cycle appears we use the edge xx^* together with Σ_0 to decompose G. Otherwise, we decompose xx^* with the cycles according to Lemmas 2.1 or 2.6. In either case we get a path decomposition of G of size at most $\frac{1}{2}u(G) + \lfloor \frac{2}{3}g(G) \rfloor$, a contradiction. □

We can assume that $N_R(x) \cap N_R(x^*) = \{a_1, a_2, \ldots, a_t\} = \{a_1^*, a_2^*, \ldots, a_t^*\}$.

Claim 3.6. *The following statements hold*:

(a) $t(xx^*)=1$ *implies* $d_R(x)=d_R(x^*)=3$.
(b) $t(xx^*)=2$ *implies* $d_R(x)=d_R(x^*)=3$ or $\{d_R(x),d_R(x^*)\}=\{3,4\}$.

Proof. From Claim 3.4 we know $d_R(x),d_R(x^*)\in\{3,4\}$. To prove part (a) we suppose $d_R(x^*)=4$. Let $G_0=G-xx^*$. Let Z be the set of even neighbors of x in G_0 and $G_1=G_0-E(x,Z)$. Let Z^* be the set of even neighbors of x^* in G_1 and $G_2=G_1-E(x^*,Z^*)$. Then $u(G_2)=u(G)+6$ and $g(G_2)\leqslant g(G)-6$. Let Σ_2 be a minimum path decomposition of G_2. Then

$$|\Sigma_2|\leqslant \tfrac{1}{2}(u(G)+6)+\lfloor\tfrac{2}{3}(g(G)-6)\rfloor\leqslant \tfrac{1}{2}u(G)+\lfloor\tfrac{2}{3}g(G)\rfloor-1.$$

Let Σ_1 be the path-cycle decomposition of G_1 obtained from Σ_2 by performing the (x^*,G_2,G_1)-construction. Then construct a path-cycle decomposition $\sum_0$ of G_0 by performing the (x,G_1,G_0)-construction. If Σ_1 contains a cycle, we can delete it before performing this construction so that Lemma 2.3 implies $|\Sigma_0|=|\Sigma_1|=|\Sigma_2|$. Observe that at most two cycles appear in Σ_0. If no cycle appears we use the edge xx^* together with Σ_0 to decompose G. Otherwise, we decompose xx^* with the cycles according to Lemmas 2.1 or 2.6. In either case we get a path decomposition of G of size at most $\frac{1}{2}u(G)+\lfloor\frac{2}{3}g(G)\rfloor$, a contradiction.

To prove part (b) we can assume $t(xx^*)=2$ and $d_R(x)=d_R(x^*)=4$. Let Z be the set of even neighbors of x except x^* and $G_1=G-E(x,Z)$. Let Z^* be the set of even neighbors of x^* in G_1 (there's only one) and $G_2=G_1-E(x^*,Z^*)$. Then $u(G_2)=u(G)+6$ and $g(G_2)\leqslant g(G)-6$. Let Σ_2 be a minimum path decomposition of G_2. Then

$$|\Sigma_2|\leqslant \tfrac{1}{2}(u(G)+6)+\lfloor\tfrac{2}{3}(g(G)-6)\rfloor\leqslant \tfrac{1}{2}u(G)+\lfloor\tfrac{2}{3}g(G)\rfloor-1.$$

Let Σ_1 be the path-cycle decomposition of G_1 obtained from Σ_2 by performing the (x^*,G_2,G_1)-construction. Although x^* is even in G_1 using Lemma 2.5 we still know that some path of Σ_1 ends at x^*. Construct a path-cycle decomposition $\sum$ of G by performing the (x^*,G_1,G)-construction. Lemma 2.3 implies $|\Sigma_1|=|\Sigma_2|$. At most one cycle appears in Σ. If no cycle appears, Σ decomposes G. Otherwise, we decompose the cycles according to Lemma 2.1. In either case we get a path decomposition of G of size at most $\frac{1}{2}u(G)+\lfloor\frac{2}{3}g(G)\rfloor$, a contradiction. □

To complete the proof we make use of the graphs Q_1,Q_2,Q_3,Q_4 shown partially in Fig. 1. These graphs are defined to be vertex induced subgraphs of R (provided they exist) on the indicated set of vertices and edges. The dotted lines indicate edges which do not exist in R (hence, not in G), and the status of all other edges must still be determined. We sometimes refer to Q_i as being an extension of xx^*. Given two graphs G,Q and a set $W\subseteq V(Q)$, we say that Q is a W-subgraph of G (written $Q\subseteq_W G$ or $G\supseteq_W Q$) if Q is a subgraph of G and $d_G(w)=d_Q(w)$ for each $w\in W$.

Claim 3.7. *At least one of the graphs Q_1,Q_2,Q_3,Q_4 is a $\{x,x^*\}$-subgraph of R.*

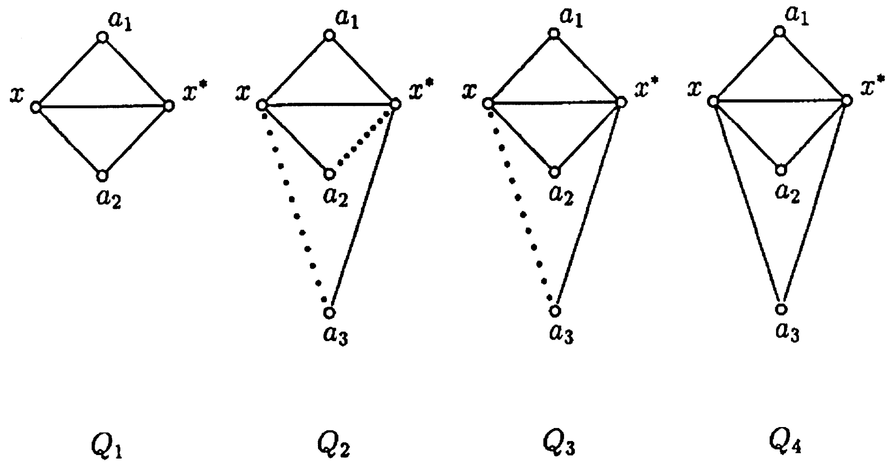

Fig. 1. Subgraphs of R used in Claim 3.7.

Proof. Since $d_R(x), d_R(x^*) \leqslant 4$, $t(xx^*) \leqslant 3$. If $t(xx^*) = 3$, we get Q_4. So Claim 3.5 leaves only the cases in Claim 3.6. Case (a) yields Q_2, and case (b) yields Q_1 if $d_R(x) = d_R(x^*) = 3$ and Q_3 if $\{d_R(x), d_R(x^*)\} = \{3, 4\}$. □

Now, we complete the description of Q_1, Q_2, Q_3, Q_4. We use K_5^- to denote the complete graph on 5 vertices with one edge deleted. If two vertex-disjoint edges are deleted, the resulting graph is called W_5.

Claim 3.8. $Q_1 = K_4$.

Proof. It suffices to show that $\bar{N}_R(V(Q_1)) = \emptyset$ because Claim 3.4 implies $d_R(a_1) \geqslant 3$ which means that $a_1a_2 \in E(R)$. Without loss of generality we can assume that a_1 has a neighbor $a^* \in V(R - Q_1)$. Since $d_R(a_1) \leqslant 4$, a_1 and a^* have at most one common neighbor. Thus, Q_2 is the only possible extension of a_1a^*, and so $d_R(a_1) = d_R(a^*) = 3$. From Claim 3.5 with a_1a^* playing the role of xx^* we see that a^* must be adjacent to x or x^* contradicting the definition of Q_1. □

Claim 3.9. $Q_2 = W_5$.

Proof. Since every edge lies in a triangle (Claim 3.5), $a_1a_2, a_1a_3 \in E(Q_2)$. Thus, $d_R(a_1) = 4$. By Claim 3.6(a) every edge of R containing a vertex of degree 4 in R lies in at least two triangles in R. So does a_1a_2. Hence, $a_2a_3 \in E(Q_2)$. If $d_R(a_2) \geqslant 4$, then for every neighbor $b \notin \{a_1, a_3, x\}$ of a_2 the edge a_2b lies in at least two triangles. So b is adjacent to x or a_1, a contradiction. Hence, $d_R(a_2) = 3$. Similarly, $d_R(a_3) = 3$. □

Claim 3.10. $Q_3 = W_5$ or K_5^-.

Proof. By Claim 3.6(a) the edge x^*a_3 lies in at least two triangles in R. Claim 3.4 implies $d_R(x^*)=4$, and so $a_1a_3, a_2a_3 \in E(Q_3)$. Thus, $t(x^*a_3)=2$ and Claim 3.6(b) implies $d_R(a_3)=3$. Similarly, $d_R(x)=3$. If $a_1a_2 \in E$, $d_R(a_1)=d_R(a_2)=4$ which gives $Q_3=K_5^-$. If $a_1a_2 \notin E$, we have $d_R(a_1)=d_R(a_2)=3$ which implies $Q_3=W_5$. □

Claim 3.11. $Q_4=K_5^-$ or K_5.

Proof. By Claim 3.6(a) the edge x^*a_1 lies in at least two triangles in R. We can assume that $a_1a_2 \in E(Q_4)$. The same argument for xa_3 allows us to include the edge a_2a_3. Thus, $d_R(a_2)=4$. If $Q_4 \neq K_5^-$ and $Q_4 \neq K_5$, we can assume that a_3 is incident with some edge of R outside Q_4 and, since this edge lies in a triangle, $d_R(a_3)>4$, a contradiction. □

Claim 3.12. *If C is a component of R, then $C \in \{K_4, W_5, K_5^-, K_5\}$.*

Proof. This follows immediately from Claims 3.7–3.11. □

Claim 3.13. *K_4 is not a component of R.*

Proof. *Assume it is.* Let $\{a,b,c,d\}$ be the set of vertices of this component, and remove the edges ab, ac, ad. Let G' be the resulting graph, and let Σ' a minimum path decomposition of G'. For each vertex $x \in \{b,c,d\}$, we have a Lovász sequence $S_x=\{x, x_1, x_2, \ldots, x'\}$. In the (a, G', G)-construction we get a path-cycle decomposition of G of cardinality $|\Sigma'|$. If no cycle is obtained, then $p(G) \leqslant p(G') \leqslant \frac{1}{2}u(G)+\lfloor \frac{2}{3}g(G) \rfloor$. Otherwise, one cycle, say C_0, is obtained, and by symmetry we can assume it contains the edges ab', ac'. This means that in the $(a, G', G-ab)$-construction we get the path decomposition $\Sigma_1=\Sigma-\{C_0\} \cup \{C_0-ab'\}$ of $G-ab$ because sequence S_b is not used. Notice that a is an end of paths $f(U_d)$ and C_0-ab', and, since a is odd in $G-ab$, a must be the end of at least one other path of Σ_1. Thus, a is an end of at least three members P, Q, S of Σ_1.

Now we perform the $(b, G-ab, G)$-construction. Since a is end of three paths of Σ_1, we have the choice between three sequences. To see that no two of them have a term in common, start by letting (p_i) and (q_j) be sequences generated by P and Q, respectively. Suppose $p_i=q_j$, and $i+j$ is minimal in satisfying this equality. We cannot have $i=1$ for otherwise $p_i=a=q_j$ would imply equality between P and Q. In general, the equality $p_i=q_j$ implies $p_ib=q_jb$ because we are using paths of a decomposition. Further, only one path contains the edge p_ib which means $p_{i-1}=q_{j-1}$, a contradiction to the minimality of $i+j$.

It follows that at most two of the sequences stop in the vertices c, d, and the third one gives rise to a path containing the edge ab. So we get a decomposition of G with the same cardinality as the one for G'. □

Claim 3.14. *Neither W_5, K_5^- nor K_5 is a component of R.*

Proof. Assume that some member C of $\{W_5, K_5^-, K_5\}$ is a component of R. Let $V(C)=\{a,b,c,d,f\}$ where a is a vertex of maximum degree in C. Let $G_1=G\cup\{fy\}$ where y is a new vertex, and let $G_1'=G\cup\{fy\}-\{ab,ac,ad\}$. Then $u(G_1')=u+6$, $g(G_1')=g-5$, and $\lambda(G_1')<\lambda(G)$. We do precisely the same constructions as in the proof of Claim 3.13, and we get the same conclusion. □

Finally, notice that Claims 3.7, 3.12–3.14 are inconsistent, and so the theorem is proved.

4. To prove the connected version

The additive constant $\frac{1}{2}$ associated with $n/2$ in Gallai's conjecture is crucial in the sense that if $\frac{1}{2}$ does not work then no constant works.

Theorem 2. *Gallai' s conjecture is equivalent to the statement that there is a constant c such that every connected graph G satisfies* $p(G)\leqslant n/2+c$.

Proof. By contradiction suppose G' is a counterexample to Gallai's conjecture, and let G be the graph obtained by starting with a new vertex and joining it to some vertex in $2k$ distinct copies of G' where $2k>2c+1$. Then

$$\frac{n}{2}+c \geqslant p(G) \geqslant 2kp(G')-k \geqslant 2k\left(\frac{n(G')}{2}+1\right)-k$$

$$= kn(G')+k=\frac{n-1}{2}+k.$$

Thus, $k-\frac{1}{2}\leqslant c$, a contradiction. □

The following conjecture is mentioned in [1] and is also a corollary of a conjecture mentioned in [10].

Conjecture 1. *If* $cy(G)=\Delta(G)/2$, *then* $p(G)\leqslant cy(G)+1$.

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